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A CENSUS OF REGULAR 3-POLYSTROMA ARISING FROM HONEYCOMBS

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A regular n-polystroma is a combinatorial structure that locally behaves like an n-dimensional regular honeycomb. We develop a list of 3-polystroma that arise naturally from 3-dimensional regular honeycombs. Most of the polystroma presented here have not appeared previously in the literature.

1. Introduction

Grünbaum [8] defines an *n-polystroma* C to be a partially ordered set with least element 0 and chains of length at most n. A *vertex* V of C is any element that is minimal in C-{0}, and the *vertex figure* of V is the polystroma consisting of V and all the elements of C which majorise V. Any maximal element F of C is called a *facet* of C and the *facet figure*, or *cell*, of F is the polystroma consisting of F and all elements of C which are majorised by F. A *flag* in C is any maximal linearly (totally) ordered subset of C. A polystroma is then said to be *regular* if its automorphism group acts transitively on its flags.

There are many models, or realizations, of polystroma; for a detailed account, the reader is referred to [8]. In that paper, Grünbaum describes 3-polystroma which arise in a natural way from the 3-dimensional hyperbolic honeycombs $\{6, 3, 3\}$ and $\{4, 4, 3\}$; he also gives a list of known polystroma of these types. Our primary aim in this paper is to extend his lists for these parameters, and to provide lists of 3-polystroma arising from the remaining 3-dimensional honeycombs. We also extend Coxeter's list [3] of twisted honeycombs (3-polystroma that can also be obtained from 3-dimensional honeycombs). Some of the polystroma in our census are already known; where possible, we give a reference. A number of the new polystroma merit detailed investigation and should suggest many interesting research problems; however, we have concentrated on providing the census in this paper. Finally, the completeness of the list is an open problem which we expect to be very hard.

In the remainder of this paper, we introduce regular maps and honeycombs,

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and describe the application of the Todd–Coxeter coset enumeration algorithm to the production of 3-polystroma. Computational results on the performance of the Todd–Coxeter technique are reported; finally, the census of 3-polystroma found is given.

2. Regular maps

A *regular tessellation* or *map* $\{p, q\}$ is a collection of p-gonal faces, q meeting at each vertex. The *Petrie polygon* of such a map is a skew polygon in which every two, but not three, consecutive edges belong to a face. Identifying all pairs of vertices separated by t steps along a Petrie polygon produces a *contracted* map, or a 2-polystroma, denoted $\{p, q\}$. For example, the cube is a spherical tessellation $\{4, 3\}$ whose Petrie polygons are hexagons. Identifying antipodal vertices produces $\{4, 3\}_3$.

The symmetry group $[p, q]$ of a regular tessellation $\{p, q\}$ is generated by the reflections R_1, R_2, R_3 in the sides P_2P_3, P_3P_1, P_1P_2 of its characteristic triangle $(p\,q\,2)$, which has angles π/p at P_3, π/q at P_1, and $\pi/2$ at P_2. It has the presentation [6]

$$R_i^2 = (R_1R_2)^p = (R_2R_3)^q = (R_3R_1)^2 = 1. \tag{2.1}$$

Transformation $R_1R_2R_3$ shifts a Petrie polygon of $\{p, q\}$ one step along itself, and the automorphism group of the map $\{p, q\}_t$ is obtained by adding the extra relation

$$(R_1R_2R_3)^t = 1$$

to (2.1) [2].

Each edge of a map has exactly two incident vertices and two incident faces. A map is *reflexible* if there is an automorphism which exchanges these two vertices without interchanging the two faces. The maps $\{p, q\}_t$ are clearly reflexible.

Let us consider the ordinary plane tessellation $\{4, 4\}$, whose vertices have integral Cartesian coordinates. By identifying all points of the lattice generated by the vectors (b, c) and $(-c, b)$, we derive the map $\{4, 4\}_{b,c}$. This can be done with any nonnegative integers, with $b \geq c$ to avoid repetition. Similarly, the infinite triangular tessellation $\{3, 6\}$ whose vertices have integral coordinates on oblique axes inclined at $\pi/3$, yields a map $\{3, 6\}_{b,c}$ derived by identifying all points of the lattice generated by the vectors (b, c) and $(-c, b+c)$. Finally, $\{6, 3\}_{b,c}$ is the dual of $\{3, 6\}_{b,c}$.

The map $\{4, 4\}_{b,c}$ is easily seen to be reflexible if and only if $bc(b-c)=0$. The groups $[4, 4]_{b,0}$ and $[4, 4]_{c,c}$ are given by (2.1) with $p=q=4$ and the respective extra relations [6]

$$(R_1R_2R_3R_2)^b = 1 \quad \text{and} \quad (R_1R_2R_3)^{2c} = 1.$$

The maps $\{3, 6\}_{b,c}$ and $\{6, 3\}_{b,c}$ are reflexible if and only if $bc(b-c)=0$. The groups $[3,6]_{b,0}$ and $[3,6]_{c,c}$ are given by (2.1) with $p=\frac{1}{2}q=3$, and the respective extra relations [6]

$$(R_1R_2R_3)^{2b}=1 \quad \text{and} \quad (R_1R_2R_3R_2R_3)^{2c}=1.$$

If b and c are different nonzero integers, the map is not reflexible. To find its symmetry group, we let $T=R_1R_3$ be the half turn about the vertex P_2 of the characteristic triangle $P_1P_2P_3$, and $S=R_2R_3$ be the rotation about the vertex P_1. The rotation subgroup of $[p,q]$ has the presentation

$$S^2=T^2=(ST)^p=1.$$

$[4,4]_{b,c}$ is generated by S and T, and has the presentation

$$S^4=T^2=(STS)^b(S^2T)^c=1.$$

In terms of the same generators, $[3,6]_{b,c}$ has the presentation

$$S^3=T^2=(ST)^6=(S^{-1}TST)^b(STS^{-1}T)^c=1.$$

The non-reflexible maps $\{p,q\}_{b,c}$ and $\{p,q\}_{c,b}$ are considered to be the same, although they are of two *enantiomorphic* (left and right) forms.

3. 3-dimensional honeycombs and associated polyhedra

A 3-dimensional regular honeycomb $\{p,q,r\}$ has cells $\{p,q\}$, r around each edge, and thus has vertex figures $\{q,r\}$. Such a honeycomb can be thought of as a 3-dimensional regular map [2]. With each $\{p,q,r\}$ we can associate a characteristic orthoscheme $P_1P_2P_3P_4$ whose dihedral angles are π/p at P_3P_4, π/q at P_4P_1, π/r at P_1P_2 and $\pi/2$ at P_1P_3, P_2P_3 and P_2P_4. The reflections R_i, in the face planes of the orthoscheme opposite to P_i, generate the symmetry group $[p,q,r]$ of $\{p,q,r\}$ with the presentation

$$R_i^2=(R_1R_2)^p=(R_2R_3)^q=(R_3R_4)^r=(R_1R_3)^2=(R_1R_4)^2=(R_2R_4)^2=1. \tag{3.1}$$

This is also the symmetry group of the honeycomb $\{r,q,p\}$, the *dual* of $\{p,q,r\}$. It is sometimes possible to derive a *contracted* honeycomb, or a 3-polystroma, by contracting each of the cells $\{p,q\}$ to $\{p,q\}_t$ or $\{p,q\}_{b,c}$. We denote such a 3-polystroma by $\{\{p,q\}_t,\{q,r\}\}$, or $\{\{p,q\}_{b,c},\{q,r\}\}$, respectively. We can similarly contract the vertex figure, or we can contract both the cells and the vertex figures. It is important to observe that $\{\{p,q\}_{b,c},\{q,r\}\}$ and $\{\{p,q\}_{c,b},\{q,r\}\}$ are essentially the same (enantiomorphic), since although their cells are of two different enantiomorphic forms, their vertex figures do not have different forms. However, $\{\{p,q\}_{b,c},\{q,r\}_{d,e}\}$ and $\{\{p,q\}_{c,b},\{q,r\}_{d,e}\}$ are not the same. Polystroma obtained by contraction of the vertex and/or cell figures are called *naturally generated* by Grünbaum [8]. The known naturally generated polystroma are listed in Table 1.

Table 1. Naturally generated 3-polystroma

Polystroma	v	e	f	c	o	Reference
$\{\{3,3\},\{3,6\}_{0,2}\}$	5	10	20	10	240	[8]
$\{\{3,3\},\{3,6\}_{0,3}\}$	12	54	108	54	1296	[8]
$\{\{3,3\},\{3,6\}_{0,4}\}$	80	640	1280	640	15360	[8]
$\{\{3,3\},\{3,6\}_{1,2}\}$	8	28	56	28	336	[8]
$\{\{3,3\},\{3,6\}_{1,3}\}$	28	182	364	182	2184	[8]
$\{\{3,3\},\{3,6\}_{1,4}\}$	64	672	1344	672	8064	
$\{\{3,3\},\{3,6\}_{2,2}\}$	20	120	240	120	2880	[8]
$\{\{3,3\},\{3,6\}_{2,3}\}$	60	570	1140	570	6840	
$\{\{3,4\}_3,\{4,4\}\}$	3	24	32	16	384	
$\{\{3,4\},\{4,4\}_{0,2}\}$	6	12	16	4	192	[8]
$\{\{3,4\},\{4,4\}_{0,3}\}$	20	90	120	30	1440	[8]
$\{\{3,4\},\{4,4\}_{1,2}\}$	6	15	20	5	120	[8]
$\{\{3,4\},\{4,4\}_{1,3}\}$	18	90	120	30	720	[8]
$\{\{3,4\},\{4,4\}_{1,4}\}$	72	612	816	204	4896	
$\{\{3,4\},\{4,4\}_{2,2}\}$	12	48	64	16	768	[8]
$\{\{3,4\},\{4,4\}_{2,3}\}$	42	273	364	91	2184	[8]
$\{\{3,5\}_5,\{5,3\}\}$	11	55	55	11	660	[5]
$\{\{3,6\},\{6,3\}_{0,2}\}$	15	60	60	5	720	
$\{\{3,6\}_{0,2},\{6,3\}_{0,2}\}$	5	20	20	5	240	[10]
$\{\{3,6\}_{0,3},\{6,3\}_{0,3}\}$	27	243	243	27	2916	[10]
$\{\{3,6\}_{0,4},\{6,3\}_{0,3}\}$	2240	20160	20160	1260	241920	
$\{\{3,6\}_{2,2},\{6,3\}_{0,3}\}$	384	3456	3456	288	41472	
$\{\{3,6\}_{1,2},\{6,3\}_{1,2}\}$	16	112	112	16	672	
$\{\{3,6\}_{4,1},\{6,3\}_{1,2}\}$	48	336	336	16	2016	
$\{\{4,3\},\{3,5\}_5\}$	64	192	240	80	3840	
$\{\{4,3\},\{3,6\}_{0,2}\}$	16	32	48	16	768	
$\{\{4,3\}_3,\{3,6\}_{0,3n}\}$	4	$18n^2$	$27n^2$	$18n^2$	$432n^2$	
$\{\{4,3\}_3,\{3,6\}_{n,n}\}$	4	$6n^2$	$9n^2$	$6n^2$	$144n^2$	
$\{\{4,3\},\{3,6\}_{1,2}\}$	48	168	252	84	2016	
$\{\{4,4\}_{n,n}\{4,4\}_{0,2}\}$	$2n^2$	$4n^2$	$4n^2$	4	$64n^2$	
$\{\{4,4\}_{0,2n},\{4,4\}_{0,2}\}$	$4n^2$	$8n^2$	$8n^2$	4	$128n^2$	
$\{\{4,4\}_{0,3},\{4,4\}_{0,3}\}$	20	90	90	20	1440	
$\{\{4,4\}_{0,4},\{4,4\}_{0,3}\}$	512	2304	2304	288	36864	
$\{\{4,4\}_{2,2},\{4,4\}_{0,3}\}$	32	144	144	36	2304	
$\{\{4,4\},\{4,4\}_{1,2}\}$	24	60	60	6	480	
$\{\{4,4\}_{1,2},\{4,4\}_{1,2}\}$	6	15	15	6	120	
$\{\{4,4\}_{1,3},\{4,4\}_{1,3}\}$	24	120	120	24	960	
$\{\{4,4\}_{3,1},\{4,4\}_{1,3}\}$	50	250	250	50	2000	
$\{\{4,4\}_{2,3},\{4,4\}_{1,3}\}$	780	3900	3900	600	31200	
$\{\{4,4\}_{2,4},\{4,4\}_{1,3}\}$	100	500	500	50	4000	
$\{\{4,4\}_{4,2},\{4,4\}_{1,3}\}$	24576	122880	122880	12288	983040	
$\{\{4,4\}_{2,2},\{4,4\}_{2,2}\}$	16	64	64	16	1024	
$\{\{4,4\}_{3,3},\{4,4\}_{2,2}\}$	144	576	576	64	9216	
$\{\{4,4\}_{2,3},\{4,4\}_{2,3}\}$	42	273	273	42	2184	
$\{\{5,3\}_5,\{3,5\}_5\}$	57	171	171	57	3420	[4]
$\{\{5,3\},\{3,6\}_{0,2}\}$	240	1200	1440	240	28800	
$\{\{5,3\}_5,\{3,6\}_{0,3}\}$	270	1215	1458	486	29160	
$\{\{6,3\}_{0,2},\{3,6\}_{0,2}\}$	10	20	20	10	480	
$\{\{6,3\}_{0,3},\{3,6\}_{0,2}\}$	54	108	108	24	2592	
$\{\{6,3\}_{0,4},\{3,6\}_{0,2}\}$	640	1280	1280	160	30720	
$\{\{6,3\}_{2,2},\{3,6\}_{0,2}\}$	120	240	240	40	5760	
$\{\{6,3\}_{1,2},\{3,6\}_{1,2}\}$	32	112	112	32	1344	
$\{\{6,3\}_{2,1},\{3,6\}_{1,2}\}$	480	1680	1680	480	20160	

For a 3-dimensional honeycomb we define a *Petrie polygon* to be a skew polygon such that every three, but no four, consecutive edges belong to a Petrie polygon of a cell [3]. In the group $[p, q, r]$, the half-turns $L = R_2R_4$, $M = R_1R_4$, and $N = R_1R_3$ generate the rotation subgroup $[p, q, r]^+$ of index two, with the presentation

$$L^2 = M^2 = N^2 = (LM)^p = (LMN)^q = (MN)^r = 1. \tag{3.2}$$

Whenever a direct symmetry group $[p, q, r]^+$ of a honeycomb has a normal subgroup generated by $(LN)^t$ and its conjugates, we can identify pairs of points separated by t steps along every right-handed Petrie polygon to obtain a regular twisted honeycomb $\{p, q, r\}_t$. Its direct symmetry group [3] is (3.2) with the addition of

$$(LN)^t = 1. \tag{3.3}$$

If the honeycomb obtained is reflexible, then the order t' of the transformation $LMNM$, which shifts a left-handed Petrie polygon one step along itself, is equal to t and the symmetry group $[p, q, r]_t$ is (3.1) with

$$(R_1R_2R_3R_4)^t = 1. \tag{3.4}$$

If, however, t' differs from t, the honeycomb is not reflexible. In this case, its symmetry group is generated by the half-turns L, M, and N and is given by (3.1) together with (3.3). The honeycomb is then said to be *chiral*. Sometimes it is possible to identify vertices of a chiral honeycomb $\{p, q, r\}_t$ separated by t steps along every left-handed Petrie polygon as well, to obtain a reflexible honeycomb $\{p, q, r\}_{t,t}$ whose symmetry group is again (3.1) with (3.4).

Let us finally remark that the addition of (3.4) to (3.1) might in some cases force the group to collapse to the identity. In other words, for some t the identification of vertices of a honeycomb will make the whole honeycomb collapse to a single point. In some other cases, the honeycomb may collapse to a single cell. At the other extreme, despite identification one might still obtain an infinite group. We exclude these cases from consideration here, focussing on cases with finite, nontrivial results.

4. The census

We present the census of 3-polystroma in three tables. We have listed one of each pair of duals, giving the number of vertices (v), the number of edges (e), the number of faces (f), and the number of cells (c). In addition, o denotes the order of the symmetry group of the polystroma. Each entry is also given with a reference; in most cases, the polystroma is new and hence no reference is supplied. Finally, it should be noted that we have omitted polystroma which are trivial.

Table 2. Chiral twisted honeycombs

Honeycomb	t'	v	e	f	c	o	Reference
$\{3,3,6\}_6$	8	8	84	168	84	1008	
$\{3,3,6\}_7$	13	14	91	182	91	1092	
$\{3,3,6\}_8$	12	32	336	672	336	4032	
$\{3,3,6\}_9$	19	114	2109	4218	2109	25308	
$\{3,4,3\}_3$	6	3	12	12	3	72	[3]
$\{3,4,4\}_4$	12	6	60	80	20	480	
$\{3,4,4\}_6$	24	72	5760	7680	1920	46080	
$\{3,4,4\}_7$	10	210	8610	11480	2870	68880	
$\{3,5,3\}_4$	5	6	60	60	6	360	[3], [9]
$\{3,5,3\}_7$	29	203	2030	2030	203	12180	
$\{3,5,3\}_9$	10	57	570	570	57	3420	
$\{3,6,3\}_4$	7	8	56	56	8	336	
$\{4,3,5\}_6$	18	114	684	855	285	6840	[3]
$\{4,3,5\}_8$	30	496	2976	3720	1240	29760	[3]
$\{4,3,6\}_4$	8	8	64	96	32	768	
$\{4,3,6\}_5$	9	60	570	855	285	6840	
$\{4,3,6\}_6$	24	192	24192	36288	12096	290304	
$\{4,4,4\}_3$	5	6	60	60	6	480	
$\{5,3,5\}_5$	9	57	342	342	57	3420	[3]
$\{5,3,6\}_4$	18	60	570	684	114	6840	
$\{6,3,6\}_3$	4	8	84	84	8	1008	
$\{6,3,6\}_4$	12	192	24192	24192	192	290304	

Table 1 provides a census of naturally generated 3-polystroma; Tables 2 and 3 list all known twisted honeycombs. The various entries in these tables were computed using an implementation of the Todd–Coxeter algorithm; in particular, we employed the lookahead method proposed in [1]. Our experience with approximately four hundred and fifty applications of this lookahead method are perhaps worth reporting, since they comprise a large test set for the algorithm and arise in a real application.

Usually, algorithms are judged by the amount of computer storage they consume, and by the amount of computing time required. These measures vary according to the size of the input, and the particular programming language and machine used. This makes exact characterizations of the efficiency of an algorithm difficult. One can estimate its practical efficiency by noting that an algorithm for coset enumeration requires space and time to list all of the cosets. A crude measure of the storage efficiency can therefore be taken to be the ratio of the maximum number of cosets ever defined to the final number of cosets. We found that this ratio varied dramatically. Approximately 80% of the time it was between 1 and 10, and thus the storage requirements were relatively modest. However, in about six percent of the cases, the ratio exceeded 100.

A crude measure of time efficiency is the ratio of the total number of cosets defined to the final number of cosets. Similar results were obtained here. The ratio lies between 1 and 10 approximately 50% of the time and exceeds 100

Table 3. Reflexible twisted honeycombs

Honeycomb	v	e	f	c	o	Reference
$\{3, 3, 3\}$	5	10	10	5	120	[3]
$\{3, 3, 4\}_4$	4	12	16	8	192	[3]
$\{3, 3, 4\}$	8	24	32	16	3186	[3]
$\{3, 3, 5\}_{15}$	60	360	600	300	7200	[3]
$\{3, 3, 5\}$	120	720	1200	600	14400	[3]
$\{3, 4, 3\}_6$	12	48	48	12	576	[3]
$\{3, 4, 3\}$	24	96	96	24	1152	[3]
$\{3, 4, 4\}_5$	10	90	120	30	1440	[7]
$\{3, 4, 4\}_{8,8}$	200	7200	9600	2400	115200	
$\{3, 5, 3\}_6$	11	110	110	11	1320	
$\{3, 6, 3\}_{5,5}$	20	320	320	20	3840	
$\{3, 6, 3\}_{6,6}$	48	576	576	48	6912	
$\{4, 3, 4\}_{3n}$	$4n^3$	$12n^3$	$12n^3$	$4n^3$	$192n^3$	[3]
$\{4, 3, 5\}_{10,10}$	64	384	480	160	7680	
$\{4, 3, 6\}_{4,4}$	8	16	24	8	384	
$\{4, 3, 6\}_{6,6}$	8	36	54	18	864	
$\{4, 3, 6\}_{7,7}$	70	420	630	210	10080	
$\{4, 4, 4\}_{5,5}$	20	180	180	20	2880	
$\{5, 3, 5\}_{6,6}$	30720	184320	184320	30720	3686400	[3]
$\{5, 3, 6\}_{6,6}$	70	1260	1512	252	30240	

approximately eight percent of the time. Typically, the time requirements of the method are not prohibitive, although they are significant.

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